The Kelly Growth Criterion

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Portfolio choice

- Playing Blackjack

Figure 1: 'Ed' Thorp

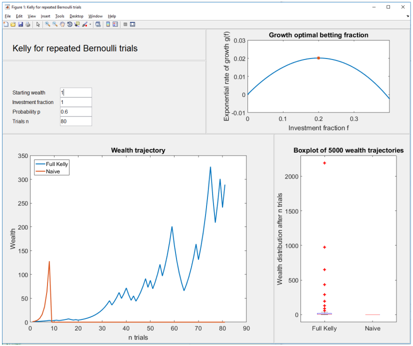

Figure 2: Matlab GUI

Portfolio choice

- Wealth for discrete returns $X_i \in \mathbb{R}^k$

$$W_n(f) = W_0 \prod_{i=1}^{n} \left(1 + \sum_{j=1}^{k} f_j X_{j,i} \right) \tag{1}$$

  - $W_0 \in \mathbb{R}^+$ starting wealth
  - $k \in \mathbb{N}^+$ assets with index j
  - $n \in \mathbb{N}^+$ periods with index i
- How to chose fraction vector $f \in \mathbb{R}^k$?

Managing Portfolio Risks

Two main strands

1. Mean-Variance approach: Markowitz (1952), Tobin (1958), Sharpe (1964) and Lintner (1965)
2. Kelly growth-optimum approach: Kelly (1956), Breiman (1961) and Thorp (1971)

Leo Breiman on BBI:

Outline

Arithmetic mean maximization

- Consider n favorable Bernoulli games with probability $\frac{1}{2} < p \leq 1$ ($q = 1 - p$) and outcome $X = 1$ (-1)
- For $\mathrm{P}(X = 1) = p = 1$, investor bets everything, $f = 1$

$$W_n = W_0 2^n \tag{2}$$

- Uncertainty - maximizing the expectation of wealth implies $f = 1$

$$\mathrm{E}(W_n) = W_0 + \sum_{i=1}^{n} (p - q)\,\mathrm{E}\left(fW_{n-1}\right), \tag{3}$$

- Leads to ruin asymptotically

$$\mathrm{P}\left(\{W_n \leq 0\}\right) = \mathrm{P}\left\{\lim_{n\to\infty} (1 - p^n)\right\} \to 1 \tag{4}$$

Minimizing risk of ruin

- Alternative: minimize the probability of ruin
- For $f = 0$

$$\mathrm{P}\left(\{W_n \leq 0\}\right) = 0 \tag{5}$$

- Minimum ruin strategy leads also to the minimization of the expected profits as no investment takes place

Geometric mean maximization

- Gambler bets a fraction of his wealth with m games won

$$W_n = W_0(1+f)^m(1-f)^{n-m} \tag{6}$$

- Exponential rate of growth per trial
(log of the geometric mean)

$$G_n(f) = \log\left(\frac{W_n}{W_0}\right)^{\frac{1}{n}} = \log\left\{(1+f)^{\frac{m}{n}}(1-f)^{\frac{n-m}{n}}\right\} \tag{7}$$

$$= \left(\frac{m}{n}\right)\log(1+f) + \left(\frac{n-m}{n}\right)\log(1-f) \tag{8}$$

Geometric mean maximization

- By Borel's law of large numbers

$$\mathrm{E}\left\{G_n(f)\right\} = g(f) = p \cdot \log(1+f) + q \cdot \log(1-f) \tag{9}$$

- Maximizing $g(f)$ w.r.t. f:

$$g'(f) = \left(\frac{p}{1+f}\right) - \left(\frac{q}{1-f}\right) = \left\{\frac{p-q-f}{(1+f)(1-f)}\right\} = 0 \tag{10}$$

$$\star\ f = f^* = p - q, \quad p \geq q > 0 \tag{11}$$

- Second derivative according to f

$$g''(f) = -\left\{\frac{p}{(1+f)^2}\right\} - \left\{\frac{q}{(1-f)^2}\right\} < 0 \tag{12}$$

Closed form for Bernoulli trials

- Growth optimal fraction, under Bernoulli trials:

$$f^* = p - q \tag{13}$$

- Maximizes the expected value of the logarithm of capital at each trial

$$\begin{aligned} g(f^*) &= p \cdot \log(1 + p - q) + q \cdot \log(1 - p - q) &(14)\\ &= p \cdot \log(p) + q \cdot \log(q) + \log(2) > 0 &(15) \end{aligned}$$

▸ A link to information theory

Bernoulli example, $p = 0.6$

- Exponential rate of asset growth for binary channel with p=0.6

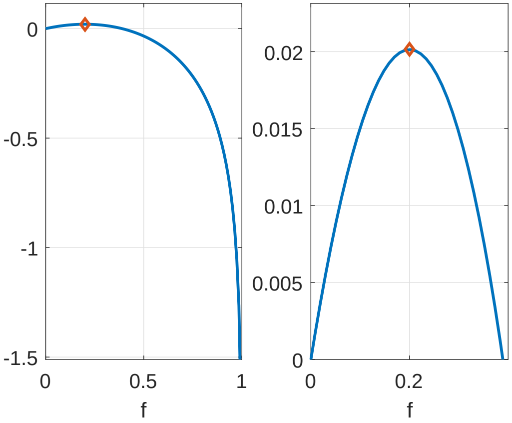

Figure 3: Bernoulli Exponential growth rate g(f)

Bernoulli

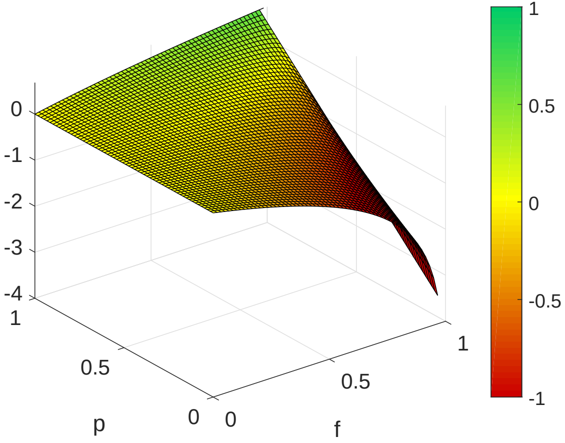

Figure 4: Bernoulli - Exponential growth rate g(f,p)

Gaussian (One-dimensional)

- $X \sim F$ with $\mathsf{E}(X) = \mu$ and $\mathsf{Var}(X) = \sigma^2$
- Return of the risk free asset $r > 0$
- Wealth given investment fractions and restriction $\sum_{j=1}^{k} f_j = 1$

$$W(f) = W_0 \{1 + (1-f)r + fX\} \tag{16}$$

$$= W_0 \{1 + r + f(X - r)\} \tag{17}$$

Gaussian (One-dimensional)

- Maximize

$$g(f) = \mathsf{E}\{\log W_n(f)\} \quad = \mathsf{E}\{G(f)\} = \mathsf{E}\log\{W_n(f)/W_0\} \tag{18}$$

- Wealth after n periods

$$W_n(f) = W_0 \prod_{i=1}^{n} \{1 + r + f(X_i - r)\} \tag{19}$$

- Taylor expansion of

$$\mathsf{E}\left[\log\left\{\frac{W_n(f)}{W_0}\right\}\right] = \mathsf{E}\left[\sum_{i=1}^{n} \log\{1 + r + f(X_i - r)\}\right] \tag{20}$$

Gaussian (One-dimensional)

- Given $\log(1+x) = x - \frac{x^2}{2} + \cdots$

$$\log\{1 + r + f(X-r)\} = r + f(X-r) - \frac{\{r + f(X-r)\}^2}{2} + \cdots \tag{21}$$

$$\approx r + f(X-r) - \frac{X^2 f^2}{2} \tag{22}$$

- Taking sum and expectation

$$\mathrm{E}\left[\sum_{i=1}^{n} \log\{1 + r + f(X_i - r)\}\right] \approx r + f(\mu_n - r) - \frac{\sigma_n^2 f^2}{2} \tag{23}$$

- Myopia: taking $\sum_{i=1}^{n} X_i$ has no impact on the solution

Gaussian (One-dimensional)

- Result of the Taylor expansion

$$g(f) = r + f(\mu - r) - \sigma^2 f^2/2 + \mathcal{O}(n^{-1/2}). \tag{24}$$

- For $n \longrightarrow \infty$, $\mathcal{O}(n^{-1/2}) \longrightarrow 0$

$$g_\infty(f) = r + f(\mu - r) - \sigma^2 f^2/2. \tag{25}$$

- Differentiating $g(f)$ according to f

$$\frac{\partial g_\infty(f)}{\partial f} = \mu - r - \sigma^2 f = 0 \Rightarrow f^* = \frac{\mu - r}{\sigma^2} = \sigma^{-1}\mathrm{MPR} \tag{26}$$

- Betting the optimal fraction f^* leads to growth rate

$$g_\infty(f^*) = \frac{(\mu - r)^2}{2\sigma^2} + r. \tag{27}$$

- $g_\infty(f)$ is parabolic around f^* with range $0 \leq f^* \leq 2f^*$

Gaussian - $\mu = 0.03, \ \sigma = 0.15, \ r = 0.01$

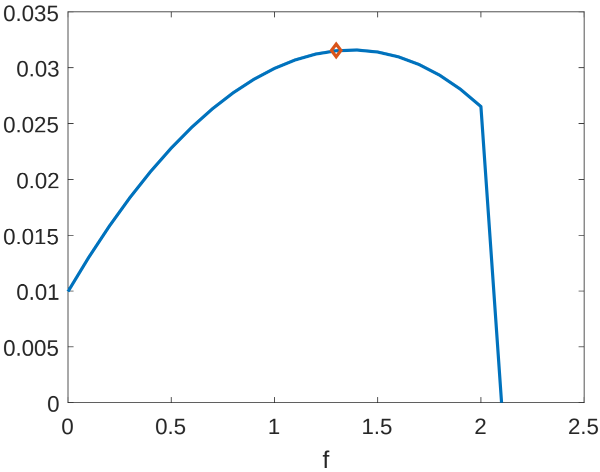

Figure 5: Gaussian approximation - Exponential growth rate g(f)

Gaussian (Multi-dimensional)

- $X \sim \mathrm{N}(\mu, \Sigma)$ and risk free rate $r > 0$

$$W_n(f) = W_0 \left\{ 1 + r + f^\top (X - r) \right\} \tag{28}$$

- Taking logarithm and expectations on both sides leads via Taylor series to

$$g(f) = \mathrm{E} \left\{ \log(1 + r) + \frac{1}{1 + r} (\mu - 1r)^\top f - \frac{1}{2(1 + r)^2} f^\top \Sigma f \right\} \tag{29}$$

- From quadratic optimization (Härdle and Simar, 2015)

$$f^* = \Sigma^{-1}(\mu - 1r) \tag{30}$$

$$g_\infty(f^*) = r + f^{*\top} \Sigma f^* / 2 \tag{31}$$

Gaussian - $\mu = [0.03\ \ 0.08],\ \ \sigma = [0.15\ \ 0.15],\ \ \rho = 0,\ \ r = 0.01$

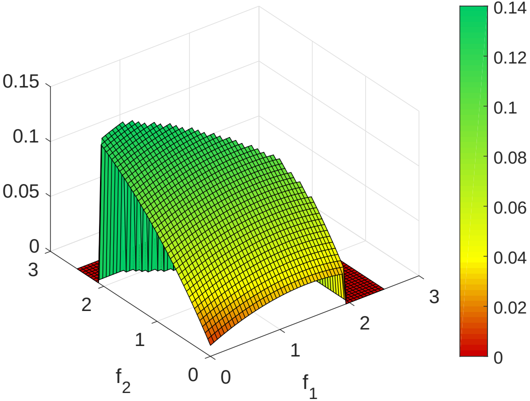

Figure 6: Gaussian approximation - Exponential growth rate g(f)

General i.i.d.

- Asymptotic dominance (in terms of wealth) of the Kelly strategy in a general i.i.d. setting in discrete time

Figure 7: Warren Buffett

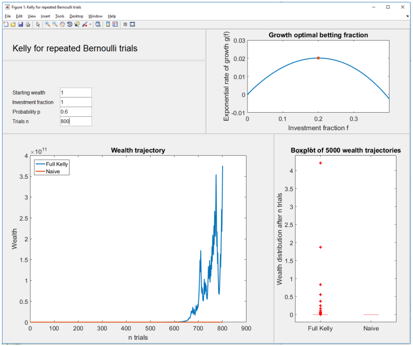

Figure 8: Matlab GUI

General i.i.d.

- Investment strategy $\Lambda = \begin{bmatrix} f_{i,j} & \cdots & f_{n,j} \\ \vdots & \ddots & \vdots \\ f_{i,k} & \cdots & f_{n,k} \end{bmatrix} = [f_i \cdots f_n]$
  - investment fractions f_i from time i to $n \in \mathbb{N}^+$
  - opportunities j to $k \in \mathbb{N}^+$
- Security price vector $p_i = \begin{bmatrix} p_{i,j} \\ \vdots \\ p_{i,k} \end{bmatrix}$
- Return per unit invested $x_i = \begin{bmatrix} \frac{p_{i,j}}{p_{i-1,j}} \\ \vdots \\ \frac{p_{i,k}}{p_{i-1,k}} \end{bmatrix}$.

Discrete i.i.d. setting

- Wealth of the investor in period n

$$W_n(f_n) = W_{n-1}(f_{n-1}) \left\{ f_n^\top x_n \right\} \tag{32}$$

- $W_n(f_n)$ increases exponentially
- Log-optimal fraction through growth rate maximization at each trial

$$f^* = \underset{f \in \mathbb{R}^k}{\operatorname{argmax}} \operatorname{E} \{\log(W_n)\} \tag{33}$$

Asymptotic outperformance

Theorem

- *Myopic log-optimal strategy* $\Lambda^* = [f^* \ \cdots \ f^*]$
- *Significantly different strategy* Λ

$$\mathrm{E}\{\log W_n(\Lambda^*)\} - \mathrm{E}\{\log W_n(\Lambda)\} \longrightarrow \infty, \qquad (34)$$

- *Kelly investor dominates asymptotically*

$$\lim_{n\to\infty} \frac{W_n(\Lambda^*)}{W_n(\Lambda)} \xrightarrow{a.s.} \infty \qquad (35)$$

Leo Breiman on BBI: 

Minimize time to reach goal g

Theorem

- *Let $N(g)$ be the smallest n, such that $W_i \geq g, \;\; g > 0$*
- *If equation (34) holds,*

$$\exists \alpha \geq 0 \perp\!\!\!\perp \Lambda, \;\; g \tag{36}$$

such that

$$\mathsf{E}\left\{N^*(g)\right\} - \mathsf{E}\left\{N(g)\right\} \leq \alpha, \tag{37}$$

- $\perp\!\!\!\perp$ *- independent of*
- Λ^* *asymptotically minimizes the time to reach goal g*

Time invariance

Theorem

- *Given a fixed set of opportunities the strategy is*
  - *fixed fraction*
  - *independent of the number of trials n*

$$\Lambda^* = [f_1^* \cdots f_n^*],\ \ f_1^* = \cdots = f_n^* \tag{38}$$

Bernoulli revisited

Theorem

- *Two investors with equal initial endowment, investment fractions f_1 and f_2*
- *For exponential growth rates*

$$G_n(f_1) > G_n(f_2) \tag{39}$$

- *the Kelly bet dominates asymptotically*

$$\lim_{n\to\infty} \frac{W_n(f_1)}{W_n(f_2)} \xrightarrow{a.s.} \infty \tag{40}$$

Bernoulli revisited

Proof.

- Difference in exponential growth rates $G_n(f) = \log\left\{\frac{W_n(f)}{W_0}\right\}^{\frac{1}{n}}$

$$\log\left\{\frac{W_n(f_1)}{W_0}\right\}^{\frac{1}{n}} - \log\left\{\frac{W_n(f_2)}{W_0}\right\}^{\frac{1}{n}} = \log\left\{\frac{W_n(f_1)}{W_n(f_2)}\right\}^{\frac{1}{n}} \tag{41}$$

- by Borel strong law of large numbers

$$\mathrm{P}\left[\lim_{n\to\infty}\log\left\{\frac{W_n(f_1)}{W_n(f_2)}\right\}^{\frac{1}{n}}\right] > 0 \xrightarrow{a.s.} 1. \tag{42}$$

Bernoulli revisited

Proof.

- For $\omega \in \Omega$, there exists $N(\omega)$ such that for $n \geq N(\omega)$,

$$W_0 \exp\{nG(f_1)\} > W_0 \exp\{nG(f_2)\} \tag{43}$$

$$W_n(f_1) > W_n(f_2) \tag{44}$$

- Asymptotically

$$\lim_{n\to\infty} \frac{W_n(f_1)}{W_n(f_2)} \xrightarrow{a.s.} \infty \tag{45}$$

□

Utility functions

- Three types of utility theories: Thorp (1971)
  - Descriptive utility - empirical data and mathematical fitting
  - Predictive utility - derives utility functions out of hypotheses
  - Normative utility - describe the behavior to achieve a certain goal
- The logarithmic utility function is used in a normative way

Conclusion

- Comparison of risk management theories
  - Markowitz-approach
    - arithmetic mean-variance efficient
    - maximizing single period returns
    - rests on two moments
  - Kelly-approach
    - geometric mean-variance efficient
    - maximize geometric rate of multi-period returns
    - utilizes the whole distribution

Information

‣ Closed form for Bernoulli trials

- Self-information (uncertainty) of outcome x

$$i(x) = -\log \mathrm{P}(x) = \log \frac{1}{\mathrm{P}(x)} \tag{46}$$

$$i(x) = 0, \ \text{for}\, \mathrm{P}(x) = 1 \tag{47}$$

$$i(x) > 1, \ \text{for}\, \mathrm{P}(x) < 1 \tag{48}$$

- Example: For a fair coin, the change of $\mathrm{P}\,(x = \{\text{tail}\}) = 0.5$

$$i(x) = -\log_2(1/0.5) = 1 \text{ bit}$$

Information

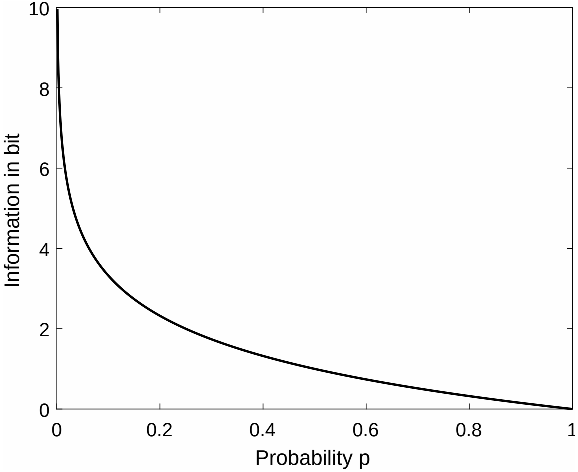

Figure 9: Self information of an outcome given probability p

Entropy

- Entropy as expectation of self-informations (average uncertainty), given outcomes $X = \{X_1, \ldots, X_n\}$

$$\mathrm{H}(X) = \mathrm{E}\{I(X)\} = -\,\mathrm{E}\{\log \mathrm{P}(X)\} \tag{49}$$

$$= -\sum_x \mathrm{P}(x) \log_2 \mathrm{P}(x) \geq 0 \tag{50}$$

- For two outcomes and $p = q = 0.5$

$$H(X) = -\,(p \log_2 p + q \log_2 q)$$
$$= -\,(1/2 \log_2 1/2 + 1/2 \log_2 1/2) = 1 \text{ bit}$$

Entropy

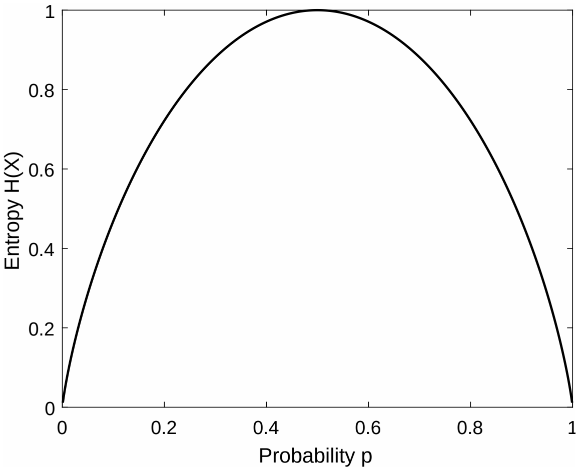

Figure 10: Entropy for two outcomes given probability p (1-p)

Entropy

- Joint entropy

$$H(X,Y) = -\,\mathrm{E}\{\log \mathrm{P}(X,Y)\} \tag{51}$$

$$= -\sum_{x,y} \mathrm{P}(x,y)\log \mathrm{P}(x,y) \tag{52}$$

- Conditional entropy

$$H(X \mid Y) = -\,\mathrm{E}\{\log \mathrm{P}(X \mid Y)\} \tag{53}$$

$$= -\sum_{x,y} \mathrm{P}(x \mid y)\log \mathrm{P}(x \mid y) \tag{54}$$

Noisy binary channel

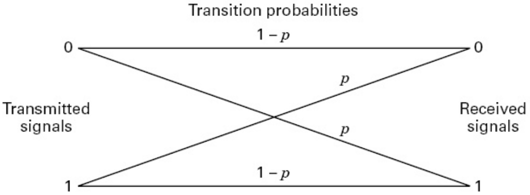

Figure 11: Noisy binary channel

Mutual information

- Mutual information

$$I(X;Y) = \mathrm{H}(X) - \mathrm{H}(X \mid Y) \tag{55}$$

$$= \mathrm{E}\left\{\log \frac{\mathrm{P}(X \mid Y)}{\mathrm{P}(X)}\right\} \tag{56}$$

- For the binary symmetric channel

$$I(X;Y) = \sum_x \sum_y \mathrm{P}(x,y) \log \frac{\mathrm{P}(x,y)}{\mathrm{P}(x)\,\mathrm{P}(y)} \tag{57}$$

$$= q\log(2q) + p\log(2p) \tag{58}$$

$$= p\log p + q\log q + \log(2) \tag{59}$$

Mutual information

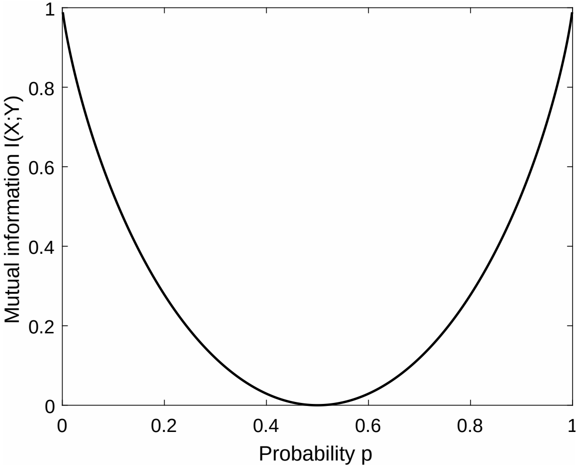

Figure 12: Mutual Information for a binary channel

Mutual information

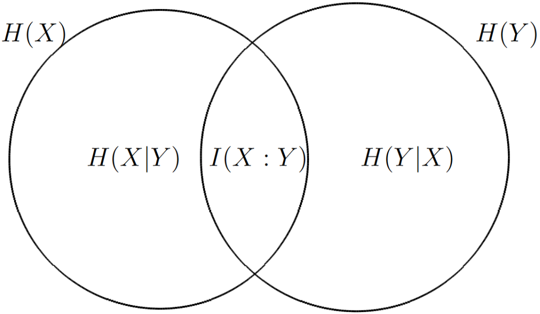

Figure 13: Relation of Entropy and Mutual Information

A link to information theory

- $I(X;Y)$ - mutual information
  - highest possible rate of information transmission in the presented channel
  - also called the channel's information carrying capacity or rate of transmission
- Equivalence to equation (14)

$$I(X;Y) = g(f^*) \tag{60}$$

▸ Closed form for Bernoulli trials

A link to estimation theory

- Relative entropy or Kullback-Leibler divergence

$$D\left(\mathrm{P}(x) \,||\, \mathrm{Q}(x)\right) = -\,\mathrm{E}\left\{\log \frac{\mathrm{P}(x)}{\mathrm{Q}(x)}\right\} \tag{61}$$

$$= \sum_{x} \mathrm{P}(x) \log \frac{\mathrm{P}(x)}{\mathrm{Q}(x)} \geq 0 \tag{62}$$

- Relation to mutual information

$$I(X;Y) = D\left\{\mathrm{P}(x,y) \,||\, \mathrm{P}(x)\,\mathrm{P}(y)\right\} \tag{63}$$

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For Further Reading

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